Sec 2.4 Composite and Inverse Functions

Composition of Functions

- For two functions $f(t)$ and $g(t)$, the function $f(g(t))$ is said to be a **composition** of f with g.
- The function $f(g(t))$ is defined by using the output of the function g as the input to f.

Given two functions f and g, the composite function, denoted by f ○ g and read as f composed with g (or f following g) is:

$$(f \circ g)(x) = f(g(x))$$

****To find the domain of any composite functions, you need to check 2 things.**

1. **g(x) or the first function to be evaluated must be defined. Where it is not is not in the domain.**

2. **then check where f(x) is not defined and set g(x) = to that.**

Ex. Given $f(x) = 3x^2 + 3$ and $g(x) = 2x$. Find and then evaluate for 2:

a. $(f \circ g)(x)$

$f(g(x)) = 3(2x)^2 + 3$
$= 3(4x^2) + 3$
$f(g(x)) = 12x^2 + 3$
$= 12(2)^2 + 3$
$= 48 + 3 = 51$

b. $(g \circ f)(x)$

$g(f(x)) = 2(3x^2+3)$
$= 6x^2 + 6$
$g(f(x)) = 6x^2 + 6$
$6(2)^2 + 6$
$24 + 6 = 30$

c. $(f \circ f)(x)$

$f(f(x)) = 3(3x^2+3)^2 + 3$
$= 3(9x^4 + 18x^2 + 9) + 3$
$= 27x^4 + 54x^2 + 27 + 3$
$f(f(x)) = 27x^4 + 54x^2 + 30$
$27(2)^4 + 54(2)^2 + 30 = 678$

d. $(g \circ g)(x)$

$g(g(x)) = 2(2x)$
$= 4x$
$g(g(x)) = 4x$
$= 4(2)$
$= 8$

Ex. Find the function and domain of $(f \circ g)(x)$ given:

$$f(x) = \frac{1}{x+2} \quad \text{and} \quad g(x) = \frac{4}{x-1}$$

g(x)
$x - 1 = 0$
$\boxed{x \neq 1}$

f(x)
$x \neq -2$
$-2 = \frac{4}{x-1}$
$-2(x-1) = 4$
$-2x + 2 = 4$
$-2x = 2$
$\boxed{x \neq -1}$

$f(g(x)) = \dfrac{1}{\frac{4}{x-1} + 2 \cdot \frac{x-1}{x-1}}$
$= \dfrac{1}{\frac{4}{x-1} + \frac{2x-2}{x-1}}$
$= \dfrac{1}{\frac{2x+2}{x-1}}$
$f(g(x)) = \dfrac{x-1}{2x+2}$

Ex. Show that $(f \circ g)(x) = (g \circ f)(x) = x$ given that $f(x) = 3x - 4$ and $g(x) = 1/3\,(x+4)$.

$f(g(x)) = 3\left(\frac{1}{3}(x+4)\right) - 4$
$= 3\left(\frac{1}{3}x + \frac{4}{3}\right) - 4$
$= x + 4 - 4$
$= x$

$g(f(x)) = \frac{1}{3}(3x - 4 + 4)$
$= \frac{1}{3}(3x)$
$= x$

Ex. Find the components f and g such that

a. $(f \circ g)(x) = (1 + x^2)^3 = f(g(x))$ $g(x) = 1 + x^2$ $f(x) = x^3$

b. $(f \circ g)(x) = \sqrt{1-x^2} = f(g(x))$ $g(x) = 1 - x^2$ $f(x) = \sqrt{x}$

Inverses of Functions: the x and y variables are reversed and the domain of the original function becomes the range of the new function while the range of the original function becomes the domain of the inverse function

Horizontal Line Test: If every horizontal line intersects the graph of a function f in at most one point, then f is one to one. (ie on a graph, if it passes the vertical line test the original equation is a function, if it passes the horizontal line test, the inverse of the original equation is a function)

Ex. Find the inverse of the following and then tell if the inverse is a function:

a. $\{(-3,-27),(-2,-3),(-1,-1),(0,0),(1,1),(2,8),(3,27)\}$
$\{(-27,-3),(-3,-2),(-1,-1),(0,0),(1,1),(8,2),(27,3)\}$ Function

b. $\{(-3,9),(-2,4),(-1,1),(0,0),(1,1),(2,4),(3,9)\}$
$\{(9,-3),(4,-2),(1,-1),(0,0),(1,1),(4,2),(9,3)\}$ Not a Function

c. $f(x) = x^2$ $x = y^2$ $\pm\sqrt{x} = y$ $f^{-1}(x) = \pm\sqrt{x}$ NOT A FUNCTION - GRAPH - $(9,3)\ (9,-3)$

d. $g(x) = x^3$ $y = x^3$ $x = y^3$ $\sqrt[3]{x} = y$ $f^{-1}(x) = \sqrt[3]{x}$ Function

Inverse function of f: Denoted by the symbol f^{-1}. To find the inverse, switch x and y and solve for y.

Theorem: If $f(f^{-1}(x)) = x$ then $f^{-1}(f(x)) = x$.

Ex. Verify that $h(x) = 3x$ and $f(x) = (1/3)x$ are inverses.

$h(f(x)) = 3(\frac{1}{3}x) = x$ $f(h(x)) = \frac{1}{3}(3x) = x$

Ex. Find the inverse of f(x) = 2x + 3. What are the domain and range of the original function and the inverse function?

$y = 2x + 3$

$x = 2y + 3$

$x - 3 = 2y$

$\frac{x}{2} - \frac{3}{2} = y$

$f^{-1}(x) = \frac{1}{2}x - \frac{3}{2}$

f(x) Domain: All Reals
Range: All Reals

$f^{-1}(x)$ Domain: All Reals
Range: All Reals

Ex. Find the inverse of $f(x) = \dfrac{2x+1}{x-1}$ where x is not equal to 1. Verify that it is the inverse.

$y = \frac{2x+1}{x-1}$

$x = \frac{2y+1}{y-1}$

$x(y-1) = 2y+1$

$xy - x = 2y + 1$

$xy - 2y = 1 + x$

$y(x-2) = 1+x$

$y = \frac{1+x}{x-2}$

$f^{-1}(x) = \frac{1+x}{x-2}$

$f(f^{-1}(x)) = \dfrac{2\left(\frac{1+x}{x-2}\right)+1}{\frac{1+x}{x-2} - 1}$

$\dfrac{\frac{2x+2}{x-2} + \frac{x-2}{x-2}}{\frac{1+x}{x-2} - \frac{(x-2)}{x-2}}$

$\dfrac{\frac{3x}{x-2}}{\frac{3}{x-2}}$

$\frac{3x}{x-2} \cdot \frac{x-2}{3}$

$\frac{3x}{3}$

$\boxed{x}$

$f^{-1}(f(x)) = \dfrac{1 + \frac{2x+1}{x-1}}{\frac{2x+1}{x-1} - 2}$

$\dfrac{\frac{x-1}{x-1} + \frac{2x+1}{x-1}}{\frac{2x+1}{x-1} - \frac{(x-1)2}{x-1}}$

$\dfrac{\frac{3x}{x-1}}{\frac{2x+1-2x+2}{x-1}}$

$\dfrac{\frac{3x}{x-1}}{\frac{3}{x-1}}$

$\frac{3x}{x-1} \cdot \frac{x-1}{3}$

$\boxed{x}$

Ex. Find the domain and range of the above problem. HINT: To find the range of the original function, find the domain of the inverse!!!

Domain: $x \neq 1$

Range: $f(x) \neq 2$

Ex. Find the inverse of $V = f(s) = s^3$ where V is volume in cubic meters and explain the meaning of:

a) f(s)
b) $f^{-1}(V)$
c.) $f^{-1}(32)$

$\sqrt[3]{V} = \sqrt[3]{s^3}$

$s = \sqrt[3]{V}$

$s = f^{-1}(V) = \sqrt[3]{V}$

a.) The volume in meters of a cube with a side length of "s" meters

b) The side length in meters of a cube with a volume of "V" cubic meters

$32 = s^3$

$\sqrt[3]{32} = s$

$3.17\text{ m} = s$

$s = \sqrt[3]{32}$

$s = 3.17\text{m}$

Do not switch the variables of application problems involving formulas! Just solve for the other variable!

HW: pg 90-92, #3-21 (m/3), 26-28, 31, 32, 35, 36, 40, 41, 42, 44, 46-49